

Construction of canonical topologies

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1. Canonical topologies

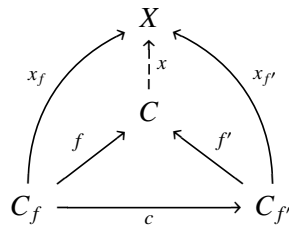
Definition 1. Let \mathcal{C} be a small category and let J be a Grothendieck topology on \mathcal{C} . J is called subcanonical, when all representable presheaves on \mathcal{C} are sheaves. J is called canonical, when it is the largest subcanonical topology.

We will show that the canonical topology does exist on any small category.

2. Effective epimorphic sieves

Definition 2. Let \mathcal{C} be a small category and let S be a sieve on C in \mathcal{C} . Consider the diagram consisting of all morphisms c such that $f' \circ c = f$ for some $f, f' \in S$. S is called effective epimorphic, when the morphisms in S form a colimit cone under this diagram.

In other words, a sieve S on C is effective epimorphic if and only if, for any family $(x_f)_{f \in S: C_f \rightarrow C}$ of morphisms $x_f: C_f \rightarrow X$ such that $x_{f'} \circ c = x_f$ holds for $f, f' \in S$ and $c: C_f \rightarrow C_{f'}$ with $f' \circ c = f$, there exists a unique factorisation x such that $x \circ f = x_f$. In a diagram,



Definition 3. Let \mathcal{C} be a small category and let S be a sieve on C in \mathcal{C} . S is called universally effective epimorphic, when g^*S is effective epimorphic for all $g: D \rightarrow C$ with codomain C .

Clearly a universally effective epimorphic sieve is effective epimorphic.

3. Construction of canonical sheaves

| Theorem 1. Any covering sieve in a subcanonical topology is universally effective epimorphic.

Consider an arbitrary representable presheaf $yX: \mathcal{C}^\circ \rightarrow \mathbf{Set}$ on a small category \mathcal{C}^{*1} . A matching family $(x_f)_{f \in S: C_f \rightarrow C}$ regarding yX consists, by definition, of morphisms $x_f: C_f \rightarrow X$ such that $x_f \circ c' = x_{f \circ c'}$ for any $f \in S$ and c' composable to f . It is equivalent to a family $(x_f)_{f \in S}$ consisting of $x_f: C_f \rightarrow X$ such that $x_{f'} \circ c = x_f$ for any $f, f' \in S$ and $c: C_f \rightarrow C_{f'}$ satisfying $f' \circ c = f$. The presheaf yX is a sheaf if and only if there exists a unique amalgamation $x: C \rightarrow X$ for any covering sieve S and any matching family $(x_f)_{f \in S}$. Since such x satisfies $x \circ f = x_f$ for $f \in S$, this exactly says that S is an effective epimorphic sieve with x being a colimit factorisation.

After all, any covering sieve S in a subcanonical topology is effective epimorphic. Moreover, for any $g: D \rightarrow C$, g^*S is also a covering sieve by the stability axiom of Grothendieck topology, which implies that g^*S is also effective epimorphic. Hence S is universally effective epimorphic.

Theorem 2. For a small category \mathcal{C} , a Grothendieck topology J defined by

$$JC := \{S \mid S \text{ is a universally effective epimorphic sieve on } C\}$$

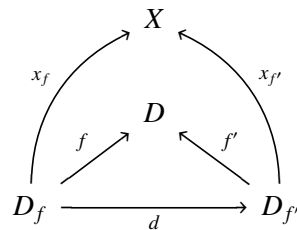
is canonical.

By the previous theorem, it suffices to show that J above forms a Grothendieck topology. Obviously J contains all maximal sieves and satisfies the stability axiom, so it remains to show that J satisfies the transitivity axiom.

Take a covering sieve $S \in JC$ and an arbitrary sieve R on C , and assume that

$$\forall s \in S \quad s^*R \in J(\text{dom } s).$$

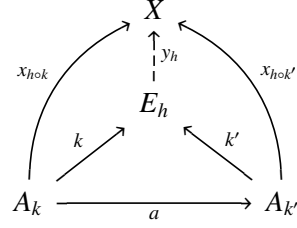
In order to prove that R is universally effective epimorphic, consider for an arbitrary $g: D \rightarrow C$ the diagram



where $f, f' \in g^*R$ are an arbitrary pair of morphisms, and d is an arbitrary morphism which makes the lower triangle commute. We will construct a morphism $x: D \rightarrow X$ such that $x \circ f = x_f$.

^{*1} Here y is the covariant Yoneda embedding, namely $yX = \text{Hom}(-, X)$.

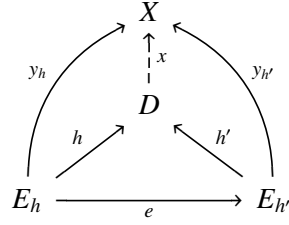
For a fixed morphism $h \in g^*S: E_h \rightarrow D$, consider the diagram



where $k, k' \in h^*g^*R$ and the lower triangle commutes. Since the outer triangle also commutes and h^*g^*R is effective epimorphic by the assumption, there exists a unique morphism $y_h: E_h \rightarrow X$ such that

$$\forall k \in h^*g^*R \quad y_h \circ k = x_{hok}.$$

Next consider



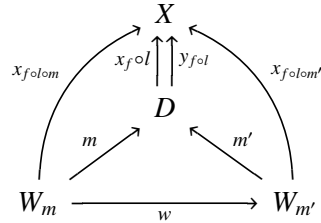
where $h, h' \in g^*S$ and the lower triangle commutes. Since $e \circ k \in h'^*g^*R$ for any $k \in h^*g^*R$,

$$\forall k \in h^*g^*R \quad (y_{h'} \circ e) \circ k = x_{h' \circ e \circ k} = x_{hok}.$$

By the uniqueness of y_h , it implies that $y_{h'} \circ e = y_h$. Thus the outer triangle in the diagram commutes, so since g^*S is effective epimorphic, there exists a unique morphism $x: D \rightarrow X$ such that

$$\forall h \in g^*R \quad x \circ h = y_h.$$

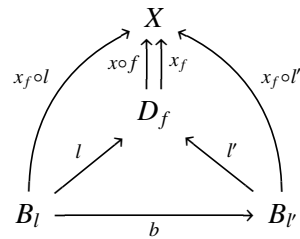
To show that x satisfies the required condition, take $f \in g^*R: D_f \rightarrow D$ and $l \in f^*g^*S: B_l \rightarrow D_f$. In the diagram



where $m, m' \in l^*f^*g^*R$, it is easily shown that $y_{f \circ l} \circ m = x_{f \circ l \circ m} = x_f \circ l \circ m$, which implies that $x_f \circ l$ and $y_{f \circ l}$ are both factorisations of the cocone formed by the lower triangle in the diagram above. This cocone is a colimit since $l^*f^*g^*R$ is effective epimorphic, so these two factorisations must coincide. After all, we obtain

$$\forall l \in f^*g^*S \quad x_f \circ l = y_{f \circ l}.$$

Then consider the diagram



with $l, l' \in f^*g^*S$. Since $x \circ f \circ l = y_{f \circ l} = x_f \circ l$, similarly by the uniqueness of factorisation, $x \circ f = x_f$. This finally shows that x is a required morphism.

References

- [1] S. MacLane, I. Moerdijk (1992) *Sheaves in Geometry and Logic*, Springer
- [2] *Subcanonical coverage in nLab* <<https://ncatlab.org/nlab/show/subcanonical+coverage>>