Construction of canonical topologies

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1. Canonical topologies

Definition 1. Let $\mathscr{C}$ be a small category and let J be a Grothendieck topology on $\mathscr{C}$. J is called subcanonical, when all representable presheaves on $\mathscr{C}$ are sheaves. J is called canonical, when it is the largest subcanonical topology.

We will show that the canonical topology does exist on any small category.

2. Effective epimorphic sieves

Definition 2. Let $\mathscr{C}$ be a small category and let S be a sieve on C in $\mathscr{C}$. Consider the diagram consisting of all morphisms c such that $f' \circ c = f$ for some $f, f' \in S$. S is called effective epimorphic, when the morphisms in S form a colimit cone under this diagram.

In other words, a sieve S on C is effective epimorphic if and only if, for any family $(x_f)_{f \in S \colon C_f \to C}$ of morphisms $x_f \colon C_f \to X$ such that $x_{f'} \circ c = x_f$ holds for $f, f' \in S$ and $c \colon C_f \to C_{f'}$ with $f' \circ c = f$, there exists a unique factorisation x such that $x \circ f = x_f$. In a diagram,

$$\begin{tikzcd}
& X & \\
& C \arrow[u, dashed, "x"'] & \\
C_f \arrow[ur, "f"] \arrow[uur, bend left, "x_f"] \arrow[rr, "c"'] & & C_{f'} \arrow[ul, "f'"'] \arrow[uul, bend right, "x_{f'}"']
\end{tikzcd}$$

Definition 3. Let $\mathscr{C}$ be a small category and let S be a sieve on C in $\mathscr{C}$. S is called universally effective epimorphic, when $g^* S$ is effective epimorphic for all $g \colon D \to C$ with codomain C.

Clearly a universally effective epimorphic sieve is effective epimorphic.

3. Construction of canonical sheaves

Theorem 1. Any covering sieve in a subcanonical topology is universally effective epimorphic.

Consider an arbitrary representable presheaf $\boldsymbol{y}X\colon \mathscr{C}^\circ \to \mathbf{Set}$ on a small category $\mathscr{C}$[*1]. A matching family $(x_f)_{f \in S\colon C_f \to C}$ regarding $\boldsymbol{y}X$ consists, by definition, of morphisms $x_f\colon C_f \to X$ such that $x_f \circ c' = x_{f \circ c'}$ for any $f \in S$ and c' composable to f. It is equivalent to a family $(x_f)_{f \in S}$ consisting of $x_f\colon C_f \to X$ such that $x_{f'} \circ c = x_f$ for any $f, f' \in S$ and $c\colon C_f \to C_{f'}$ satisfying $f' \circ c = f$. The presheaf $\boldsymbol{y}X$ is a sheaf if and only if there exists a unique amalgamation $x\colon C \to X$ for any covering sieve S and any matching family $(x_f)_{f \in S}$. Since such x satisfies $x \circ f = x_f$ for $f \in S$, this exactly says that S is an effective epimorphic sieve with x being a colimit factorisation.

After all, any covering sieve S in a subcanonical topology is effective epimorphic. Moreover, for any $g\colon D \to C$, g^*S is also a covering sieve by the stability axiom of Grothendieck topology, which implies that g^*S is also effective epimorphic. Hence S is universally effective epimorphic.

Theorem 2. For a small category $\mathscr{C}$, a Grothendieck topology J defined by

$$JC \coloneqq \{S \mid S \text{ is a universally effective epimorphic sieve on } C\}$$

is canonical.

By the previous theorem, it suffices to show that J above forms a Grothendieck topology. Obviously J contains all maximal sieves and satisfies the stability axiom, so it remains to show that J satisfies the transitivity axiom.

Take a covering sieve $S \in JC$ and an arbitrary sieve R on C, and assume that

$$^\forall s \in S \quad s^*R \in J(\operatorname{dom} s).$$

In order to prove that R is universally effective epimorphic, consider for an arbitrary $g\colon D \to C$ the diagram

$$\begin{tikzcd}
 & X & \\
 & D & \\
D_f \arrow[uur, bend left, "x_f"] \arrow[ur, "f"] \arrow[rr, "d"'] & & D_{f'} \arrow[uul, bend right, "x_{f'}"'] \arrow[ul, "f'"']
\end{tikzcd}$$

where $f, f' \in g^*R$ are an arbitrary pair of morphisms, and d is an arbitrary morphism which makes the lower triangle commute. We will construct a morphism $x\colon D \to X$ such that $x \circ f = x_f$.

[*1] Here $\boldsymbol{y}$ is the covariant Yoneda embedding, namely $\boldsymbol{y}X = \mathrm{Hom}(\text{-}, X)$.

For a fixed morphism $h \in g^*S\colon E_h \to D$, consider the diagram

$$\begin{tikzcd}
 & X & \\
 & E_h \arrow[u, dashed, "y_h"'] & \\
A_k \arrow[uur, bend left, "x_{h\circ k}"] \arrow[ur, "k"] \arrow[rr, "a"'] & & A_{k'} \arrow[uul, bend right, "x_{h\circ k'}"'] \arrow[ul, "k'"']
\end{tikzcd}$$

where $k, k' \in h^*g^*R$ and the lower triangle commutes. Since the outer triangle also commutes and h^*g^*R is effective epimorphic by the assumption, there exists a unique morphism $y_h\colon E_h \to X$ such that

$$^\forall k \in h^*g^*R \;\; y_h \circ k = x_{h\circ k}.$$

Next consider

$$\begin{tikzcd}
 & X & \\
 & D \arrow[u, dashed, "x"'] & \\
E_h \arrow[uur, bend left, "y_h"] \arrow[ur, "h"] \arrow[rr, "e"'] & & E_{h'} \arrow[uul, bend right, "y_{h'}"'] \arrow[ul, "h'"']
\end{tikzcd}$$

where $h, h' \in g^*S$ and the lower triangle commutes. Since $e \circ k \in h'^*g^*R$ for any $k \in h^*g^*R$,

$$^\forall k \in h^*g^*R \;\; (y_{h'} \circ e) \circ k = x_{h'\circ e\circ k} = x_{h\circ k}.$$

By the uniqueness of y_h, it implies that $y_{h'} \circ e = y_h$. Thus the outer triangle in the diagram commutes, so since g^*S is effective epimorphic, there exists a unique morphism $x\colon D \to X$ such that

$$^\forall h \in g^*R \;\; x \circ h = y_h.$$

To show that x satisfies the required condition, take $f \in g^*R\colon D_f \to D$ and $l \in f^*g^*S\colon B_l \to D_f$. In the diagram

$$\begin{tikzcd}
 & X & \\
 & D \arrow[u, shift right, "x_f\circ l"] \arrow[u, shift left, "y_{f\circ l}"'] & \\
W_m \arrow[uur, bend left, "x_{f\circ l\circ m}"] \arrow[ur, "m"] \arrow[rr, "w"'] & & W_{m'} \arrow[uul, bend right, "x_{f\circ l\circ m'}"'] \arrow[ul, "m'"']
\end{tikzcd}$$

where $m, m' \in l^*f^*g^*R$, it is easily shown that $y_{f\circ l} \circ m = x_{f\circ l\circ m} = x_f \circ l \circ m$, which implies that $x_f \circ l$ and $y_{f\circ l}$ are both factorisations of the cocone formed by the lower triangle in the diagram above. This cocone is a colimit since $l^*f^*g^*R$ is effective epimorphic, so these two factorisations must coincide. After all, we obtain

$$^\forall l \in f^*g^*S \;\; x_f \circ l = y_{f\circ l}.$$

Then consider the diagram

$$
\begin{tikzcd}
& X & \\
& D_f \arrow[u, shift right, "x\circ f"] \arrow[u, shift left, "x_f"'] & \\
B_l \arrow[ur, "l"] \arrow[uur, bend left, "x_f\circ l"] \arrow[rr, "b"'] & & B_{l'} \arrow[ul, "l'"'] \arrow[uul, bend right, "x_f\circ l'"']
\end{tikzcd}
$$

with $l, l' \in f^*g^*S$. Since $x \circ f \circ l = y_{f\circ l} = x_f \circ l$, similarly by the uniqueness of factorisation, $x \circ f = x_f$. This finally shows that x is a required morphism.

References

[1] S. MacLane, I. Moerdijk (1992) *Sheaves in Geometry and Logic*, Springer

[2] *Subcanonical coverage in nLab* <https://ncatlab.org/nlab/show/subcanonical+coverage>